Geometric Quantum Gates, Composite Pulses, and Trotter-Suzuki Formulas

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Introduction

composite quantum gates

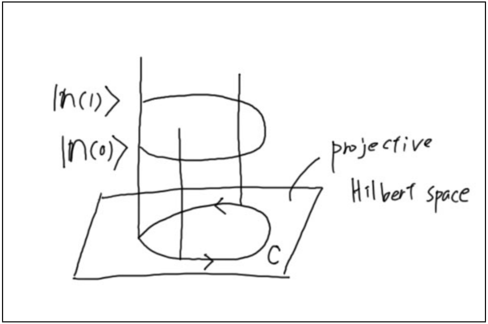
$$\prod_{j=1}^{N} R(\boldsymbol{m}_j, \theta_j) = R(\boldsymbol{m}, \theta)$$

fully compensating composite quantum gate

$$\prod_{j=1}^{N} R(\boldsymbol{m}_j, \theta_j(1+\varepsilon)) = R(\boldsymbol{m}, \theta) + \underline{O(\varepsilon^2)}$$

Geometric Quantum Gates

phases



$$|\boldsymbol{n}(1)\rangle = \underline{e^{i\gamma}}\,|\boldsymbol{n}(0)\rangle \quad , \quad \gamma = \gamma_{\rm d} + \gamma_{\rm g}$$

$\gamma_{\rm d}$: dynamic phase , $\gamma_{\rm g}$: geometric phase

Geometric Quantum Gates

geometric phase: $$\gamma_{\rm g} = i \int_0^1 \langle \boldsymbol{n}(t)| \frac{d}{dt} |\boldsymbol{n}(t)\rangle \, dt$$

dynamic phase: $$\gamma_{\rm d} = - \int_0^1 \langle \boldsymbol{n}(t)| H |n(t)\rangle \, dt$$

H: Hamiltonian

Composite Pulse

single-qubit operations

$$R(\boldsymbol{m}, \theta) = \exp\left(-i\theta \frac{\boldsymbol{m} \cdot \boldsymbol{\sigma}}{2}\right)$$

θ: control field strength , $\boldsymbol{m}$: axis of rotation ($|\boldsymbol{m}| = 1$)

$$\boldsymbol{\sigma} = (\sigma_x,\ \sigma_y,\ \sigma_z)$$

error in θ

$$R(\boldsymbol{m}, \theta(1+\varepsilon)) = R(\boldsymbol{m}, \theta) + O(\varepsilon)$$

Composite Pulse

single-qubit operations

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Composite Pulse

composite quantum gates

$$\prod_{j=1}^{N} R(\boldsymbol{m}_j, \theta_j) = R(\boldsymbol{m}, \theta)$$

fully compensating composite quantum gate

$$\prod_{j=1}^{N} R(\boldsymbol{m}_j, \theta_j(1+\varepsilon)) = R(\boldsymbol{m}, \theta) + \underline{O(\varepsilon^2)}$$

Composite Pulse

composite quantum gates

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fully compensating composite quantum gate

$$\prod_{j=1}^{N} R(\boldsymbol{m}_j, \theta_j(1+\varepsilon)) = R(\boldsymbol{m}, \theta) + \underline{O(\varepsilon^2)}$$

Composite Pulse

single-qubit operation

Hamiltonian

$$R(\boldsymbol{m},\theta) = \exp\left(-i\frac{\theta}{2}\boldsymbol{m}\cdot\boldsymbol{\sigma}\right) \quad \Longleftrightarrow \quad H(\boldsymbol{m},\theta) = \frac{\theta}{2}\boldsymbol{m}\cdot\boldsymbol{\sigma}$$

dynamic phase

$$\gamma_{\mathrm{d}} = -\int_0^1 \langle \boldsymbol{n}|\, H(\boldsymbol{m},\theta)\, |\boldsymbol{n}\rangle = -\frac{\theta}{2}\boldsymbol{m}\cdot\boldsymbol{n}$$

$\boldsymbol{n}$: Bloch vector , $|\boldsymbol{n}\rangle$: state vector

$$\boldsymbol{n} = \langle \boldsymbol{n}|\, \boldsymbol{\sigma}\, |\boldsymbol{n}\rangle$$

Composite Pulse

single-qubit operation

Hamiltonian

$$R(\boldsymbol{m},\theta) = \exp\left(-i\frac{\theta}{2}\boldsymbol{m}\cdot\boldsymbol{\sigma}\right) \quad \Longleftrightarrow \quad H(\boldsymbol{m},\theta) = \frac{\theta}{2}\boldsymbol{m}\cdot\boldsymbol{\sigma}$$

dynamic phase

$$\underline{\gamma_{\mathrm{d}} = -\int_0^1 \langle \boldsymbol{n}| H(\boldsymbol{m},\theta) |\boldsymbol{n}\rangle = -\frac{\theta}{2}\boldsymbol{m}\cdot\boldsymbol{n}}$$

$\boldsymbol{n}$: Bloch vector , $|\boldsymbol{n}\rangle$: state vector

$$\boldsymbol{n} = \langle \boldsymbol{n}| \boldsymbol{\sigma} |\boldsymbol{n}\rangle$$

Composite Pulse and Dynamic Phase

systematic control field strength error

$$\prod_{j=1}^{N} R(\boldsymbol{m}_j, \theta_j(1+\varepsilon)) = R(\boldsymbol{m}, \theta) + \sum_{j=1}^{N} R_N \dots R_j \left(-i\varepsilon \frac{\theta_j}{2} \boldsymbol{m}_j \cdot \boldsymbol{\sigma}\right) \dots R_1 + O(\varepsilon^2)$$

$$= R(\boldsymbol{m}, \theta) - \underline{i\varepsilon \sum_{j=1}^{N} R_N \dots (R_j H_j) \dots R_1} + O(\varepsilon^2)$$

$$R_j = R(\boldsymbol{m_j}, \theta_j) = \exp\left(-i\theta_j \frac{\boldsymbol{m}_j \cdot \boldsymbol{\sigma}}{2}\right)$$

$$H_j = H(\boldsymbol{m}_j, \theta_j) = \frac{\theta_j}{2} \boldsymbol{m}_j \cdot \boldsymbol{\sigma}$$

Composite Pulse and Dynamic Phase

expectation value

$$-\langle \boldsymbol{n}_0 | \sum_{j=1}^{N} R_N \dots R_j H_j R_{j-1} \dots R_1 |\boldsymbol{n}_0\rangle = -e^{-i\theta/2} \sum_{j=1}^{N} \langle \boldsymbol{n}_{j-1} | H_j |\boldsymbol{n}_{j-1}\rangle$$

$$= -e^{-i\theta/2} \sum_{j=1}^{N} \gamma_{\mathrm{d},j}$$

$$\sum_{j=1}^{N} \gamma_{\mathrm{d},j} = 0 \quad \Longrightarrow \quad \prod_{j=1}^{N} R(\boldsymbol{m}_j, \theta_j(1+\varepsilon)) = \underline{R(\boldsymbol{m}, \theta) + O(\varepsilon^2)}$$

composite quantum gate is robust against control field strength error.

SCROFULOUS pulse

SCROFULOUS pulse

$$R(\boldsymbol{m}_1, \theta_1) R(\boldsymbol{m}_2, \pi) R(\boldsymbol{m}_1, \theta_1) = R(\boldsymbol{x}, \theta)$$

$$\boldsymbol{m}_i = (\cos\phi_i,\ \sin\phi_i,\ 0)$$

SCROFULOUS pulse

condition 1

$$\langle \boldsymbol{x}|\, R(\boldsymbol{m}_1,\theta_1)R(\boldsymbol{m}_2,\pi)R(\boldsymbol{m}_1,\theta_1)\,|\boldsymbol{x}\rangle = e^{-i\theta/2}$$

$$\Longrightarrow \quad \cos\theta_1 = \frac{\tan(\phi_1-\phi_2)}{\tan\phi_1} \quad , \quad \sin\frac{\theta}{2} = \frac{\sin(\phi_1-\phi_2)}{\sin\phi_1}$$

condition 2

$$\sum_{j=1}^{3} \langle \boldsymbol{n}_{j-1}|\, H_j\, |\boldsymbol{n}_{j-1}\rangle = 0$$

$$\Longrightarrow \quad 2\theta_1\cos(\phi_1-\phi_2)+\pi = 0$$

same as H. K. Cummins, G. Llewellyn, and J. A. Jones, Phys. Rev. A **67**, 042308 (2003).

SCROFULOUS pulse

condition 1

$$\langle \boldsymbol{x} | R(\boldsymbol{m}_1, \theta_1) R(\boldsymbol{m}_2, \pi) R(\boldsymbol{m}_1, \theta_1) | \boldsymbol{x} \rangle = e^{-i\theta/2}$$

$$\Longrightarrow \quad \cos\theta_1 = \frac{\tan(\phi_1 - \phi_2)}{\tan\phi_1} \quad , \quad \sin\frac{\theta}{2} = \frac{\sin(\phi_1 - \phi_2)}{\sin\phi_1}$$

condition 2

$$\sum_{j=1}^{3} \langle \boldsymbol{n}_{j-1} | H_j | \boldsymbol{n}_{j-1} \rangle = 0$$

$$\Longrightarrow \quad 2\theta_1 \cos(\phi_1 - \phi_2) + \pi = 0$$

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Trotter-Suzuki Formula

Trotter-Suzuki formula

when $B = \sum A_l$,

$$\exp\left(-iBt\right) = \prod_l \exp\left(-iA_l t\right) + O(t^2)$$

A: series of Hamiltonian

B: Hamiltonian

Composite Pulse based on Trotter-Suzuki Formula

$$R(\boldsymbol{x},\, \theta)$$

erroneous rotation

$$R(\boldsymbol{x},\, \theta(1+\varepsilon)) = R(\boldsymbol{x},\, \theta)\underline{R(\boldsymbol{x},\, \theta\varepsilon)}$$

if $\prod_j R_j = R(\boldsymbol{x},\, -\theta\varepsilon)$,

$$\left(\prod_j R_j\right) R(\boldsymbol{x},\, \theta) \;\longrightarrow\; \text{reliable quantum gate}$$

Composite Pulse based on Trotter-Suzuki Formula

Trotter-Suzuki formula

$$\exp\left(-iBt\right) = \prod_{l} \exp\left(-iA_l t\right) + O(t^2)$$

$$R(\boldsymbol{x},\, -\theta\varepsilon) = \exp\left[-i\left(-\theta\frac{\sigma_x}{2}\right)\varepsilon\right]$$

$$= \prod_{l=1}^{N} \exp\left(-iA_l\varepsilon\right) + O(\varepsilon^2)$$

where

$$\sum_{l=1}^{N} A_l = -\theta\frac{\sigma_x}{2}$$

Composite Pulse based on Trotter-Suzuki Formula

Now we set $N = 2$.

$$A_l \;\longrightarrow\; A_+,\; A_-$$

For example,

$$A_\pm = -2\pi \frac{\boldsymbol{m}_\pm \cdot \boldsymbol{\sigma}}{2}$$

$$\boldsymbol{m}_\pm = (\cos\phi,\; \pm\sin\phi,\; 0)\;, \qquad \phi = \cos^{-1}\left(\frac{\theta}{4\pi}\right)$$

$$\Downarrow$$

$$R(\boldsymbol{m}_+,\, -2\pi(1+\varepsilon))R(\boldsymbol{m}_-,\, -2\pi(1+\varepsilon)) = \underline{\exp(-iA_+\varepsilon)\exp(-iA_-\varepsilon)}$$

$$\boxed{R(\boldsymbol{m}_+,\, -2\pi)R(\boldsymbol{m}_-,\, -2\pi)R(\boldsymbol{x},\, \theta) \implies \text{reliable quantum gate}}$$

Composite Pulse based on Trotter-Suzuki Formula

condition

$$A_+ + A_- = -\theta\frac{\sigma_x}{2}$$

$$\begin{aligned} 0 &= \theta\frac{\sigma_x}{2} + A_+ + A_- \\ &= \langle \boldsymbol{x}|\,\theta\frac{\sigma_x}{2} + A_+ + A_-\,|\boldsymbol{x}\rangle \\ &= \langle \boldsymbol{x}|\,\theta\frac{\sigma_x}{2}\,|\boldsymbol{x}\rangle + \langle \boldsymbol{x}|\,R(\boldsymbol{x},\,-\theta)A_-R(\boldsymbol{x},\,\theta)\,|\boldsymbol{x}\rangle \\ &\qquad + \langle \boldsymbol{x}|\,R(\boldsymbol{x},\,-\theta)R(\boldsymbol{m}_-,\,2\pi)A_+R(\boldsymbol{m}_-,\,-2\pi)R(\boldsymbol{x},\,-\theta)\,|\boldsymbol{x}\rangle \end{aligned}$$

condition for using Trotter-Suzuki formura

$$A_+ + A_- = -\theta\frac{\sigma_x}{2}$$

$=$

$$\sum_j \gamma_{\mathrm{d}j} = 0$$

Summary

- All GQG with Aharanov-Anandan phases are fully compensating composite quantum gates that are robust against control field stlength errors.

- The condition for using the Trotter-Suzuki formuras is equivalent that the sum of dynamic phases in a composite quantum gate is vanishing.