Holonomic Quantum Gates

using

Isospectral Deformations of Ising Model

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**Goal**

Implementation of a holonomic quantum computing in a physical system

- interesting mathematically
- expected to be robust against noise

- With a spin-chain model (proposal with a realistic system)

- Nuclear spin
- Logical qubit
- Quantum gate
- Holonomy

Interaction: Heisenberg-type → Other type of interaction ??

In this work: with Ising-type interaction, Liquid-state NMR in mind !
Holonomic Quantum Computing

- What is holonomy?

\[ u(1) = u(0) \cdot \Gamma[C] \]

\( \Gamma \): Holonomy associated with anti-Hermitian connection

- What is Holonomic Quantum Computing?

  Quantum computing using holonomy

Isospectral deformation of Hamiltonian

Isospectrally deforming Hamiltonian

Time-dependent Hamiltonian as follows

\[ H(t/T) = e^{Xt/T} H_0 e^{-Xt/T} \quad (0 \leq t \leq T) \]

How to choose \( X \)

- It is an anti-Hermitian operator.
- At \( t = T \), the Hamiltonian returns to the initial one.
- No undesired transitions into the non-coding space exist (described later).
Holonomy associated with anti-Hermitian connection

adiabatic approximation

\[
T \left[ \exp\left(-i \int_0^T H(t/T) \, dt \right) \right] P_0 \approx e^{-i \int_0^T E_0(t/T) \, dt} \Gamma P_0
\]

\[
P_0 = \sum_{i=1}^{g} |E_0, i\rangle \langle E_0, i| : \text{projection operator onto the subspace spaned by the ground states}
\]

\[
H(t/T) = e^{Xt/T} H_0 e^{-Xt/T}
\]

Isospectral deformation

\[
A_{ij}(\tau) = A_{ij} = \langle E_0, i | X | E_0, j \rangle : \text{(element of) anti-Hermitian connection}
\]

\[
\Rightarrow \quad \Gamma = T \left[ \exp\left(-\int_0^1 A(\tau) \, d\tau \right) \right] = e^{-A}
\]

The time–evolution operator is determined by \( A \).
Hamiltonian of two-spin system with Ising-type interaction

- Hamiltonian

\[ H_0 = -\omega \sigma_z \otimes I - \omega I \otimes \sigma_z + J \sigma_z \otimes \sigma_z \]

\[ |T_+\rangle = |++\rangle, \quad |T_0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |--\rangle), \quad |T_-\rangle = |--\rangle, \quad |S_0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |--\rangle) \]

\[ H_0 |T_+\rangle = (-2\omega + J) |T_+\rangle \]
\[ H_0 |T_0\rangle = -J |T_0\rangle \]
\[ H_0 |T_-\rangle = (2\omega + J) |T_-\rangle \]
\[ H_0 |S_0\rangle = -J |S_0\rangle \]

\[ \omega = J \quad \Rightarrow \quad \text{3-fold degenerate ground states} \]

\[ |T_+\rangle, \quad |T_0\rangle, \quad |S_0\rangle \]
Result : One-Qubit Quantum Gate

\[ X = i n \cdot (\Omega_1 \sigma_1 + \Omega_2 \sigma_2) \]

taking \( \Omega_1 = \Omega_2 = \nu \)

\[
A = \begin{pmatrix}
|T_+\rangle & |T_0\rangle & |S_0\rangle \\
2i\nu n_z & \sqrt{2}i\nu (n_x - in_y) & 0 \\
\sqrt{2}i\nu (n_x + in_y) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
A \downarrow_{(T_+, T_0)} = i \begin{pmatrix}
|T_+\rangle & |T_0\rangle \\
2\nu n_z & \sqrt{2}\nu (n_x - in_y) \\
\sqrt{2}\nu (n_x + in_y) & 0
\end{pmatrix}
\]

\[
= i [\nu n_z (I + \sigma_z) + \sqrt{2}\nu (n_x \sigma_x + n_y \sigma_y)]
\]

\[
\sigma_1 = \sigma \otimes I , \quad \sigma_2 = I \otimes \sigma \\
n = (n_x, n_y, n_z) \\
\sigma = (\sigma_x, \sigma_y, \sigma_z), \quad |n| = 1
\]

There is no transition between \( |T_+\rangle, |T_0\rangle \) and \( |S_0\rangle \)

\[
\downarrow \\
|T_+\rangle, |T_0\rangle \rightarrow |0\rangle, |1\rangle
\]

\[
|T_-\rangle \quad \text{excited state} \\
|T_+\rangle |T_0\rangle |S_0\rangle \quad \text{ground state} \\
4J
\]
taking $\nu = \kappa$ \(\pi \ (\kappa \in \mathbb{N})\) 

closed loop in the parameter space

$$\Gamma = e^{-i\kappa\pi n_z} e^{-i\kappa\pi \sqrt{2-n_z^2}} m \cdot \sigma$$

$$= e^{-i\pi n_z} \begin{pmatrix} \cos \theta - i m_z \sin \theta & -(i m_x + m_y) \sin \theta \\ (-i m_x + m_y) \sin \theta & \cos \theta - i m_z \sin \theta \end{pmatrix}$$

$$\theta = \kappa \pi \sqrt{2-n_z^2}, \quad m = \left( \frac{\sqrt{2}n_x}{\sqrt{2-n_z^2}}, \frac{\sqrt{2}n_y}{\sqrt{2-n_z^2}}, \frac{n_z}{\sqrt{2-n_z^2}} \right), \quad |m| = 1$$
Construction of Quantum Gates

Example

taking \( n_1 = (1, 0, 0) \), \( n_2 = (0, 1, 0) \), \( \kappa = 1 \),

\[
\Gamma_1 = \begin{pmatrix}
\cos(\sqrt{2}\pi) & -i\sin(\sqrt{2}\pi) \\
-i\sin(\sqrt{2}\pi) & \cos(\sqrt{2}\pi)
\end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix}
\cos(\sqrt{2}\pi) & -\sin(\sqrt{2}\pi) \\
\sin(\sqrt{2}\pi) & \cos(\sqrt{2}\pi)
\end{pmatrix}, \quad [\Gamma_1, \Gamma_2] \neq 0
\]
Hadamard gate
taking
\[ m = \frac{1}{\sqrt{2}}(1, 0, 1), \quad \frac{\theta_H}{\pi} = \frac{2}{\sqrt{3}} \kappa, \quad \sigma_H = \frac{1}{\sqrt{2}}(\sigma_{Lx} + \sigma_{Lz}) \]

\[ \frac{\theta_H}{\pi} = \frac{1}{2} + 2n\pi \]

\( \kappa \) cannot fulfill this condition.

Here, the Hadamard gate could be constructed if the following condition would be satisfied,

\[ |(\theta - \theta_{\kappa}) \mod 2\pi| < \varepsilon. \]
Result 2: Controlled Gate

\[ H_{2D} = H^{(12)} \otimes 1^{(34)} + 1^{(12)} \otimes H^{(34)} \]

\[ X = X^{(12)} + X^{(34)} + X^{(12)-(34)} \]

The Hamiltonian should return to the initial one at the end.

\[ H^{(12)} = - J_{12} \sigma_{1z} + J_{12} \sigma_{2z} + J_{12} \sigma_{1z} \sigma_{2z} \]
\[ H^{(34)} = - J_{34} \sigma_{3z} - J_{34} \sigma_{4z} + J_{34} \sigma_{3z} \sigma_{4z} \]

\[ X^{(12)} = i n_{(12)} \cdot (\Omega_{(12)} \sigma_{1} + \Omega_{(12)} \sigma_{2}) \]
\[ X^{(34)} = i n_{(34)} \cdot (\Omega_{(34)} \sigma_{3} + \Omega_{(34)} \sigma_{4}) \]
\[ X^{(12)-(34)} = i J(\sigma_{1z} \sigma_{3z} + \sigma_{1z} \sigma_{4z} + \sigma_{2z} \sigma_{3z} + \sigma_{2z} \sigma_{4z}) \]

The obtained gate is

\[ \Gamma^{(2)}(J, \kappa') = (-1)^{\kappa'} \Gamma^{L}(J, \kappa') \Gamma^{C}(J), \quad \text{where} \]

\[ \Gamma^{L}(J, \kappa') = e^{-i(\kappa' \pi + J) \sigma_{Lz}} \otimes e^{-i\kappa L_{z} \sigma_{Lz}}, \quad \Gamma^{C}(J) = |0\rangle_{L} \langle 0| \otimes I_{L} + |1\rangle_{L} \langle 1| \otimes e^{i2J \sigma_{Lz}}. \]
Summary

Two-spin system (Ising-type interaction)

Construct the logical qubit

- Two one-qubit gates that are non-commutable with each other
- Controlled-Z-rotation gate
- Approximate Hadamard gate

Analytical calculation of the holonomy