Holonomic Quantum Gates using Isospectral Deformations of Ising Model

Symposium on Decoherence Suppression in Quantum Systems

Masamitsu Bando[A]

Yukihiro Ota[B], Yasusi Kondo[A], Mikio Nakahara[A]

Department of Physics, Kinki University[A],
Research Center for Quantum Computing[B]

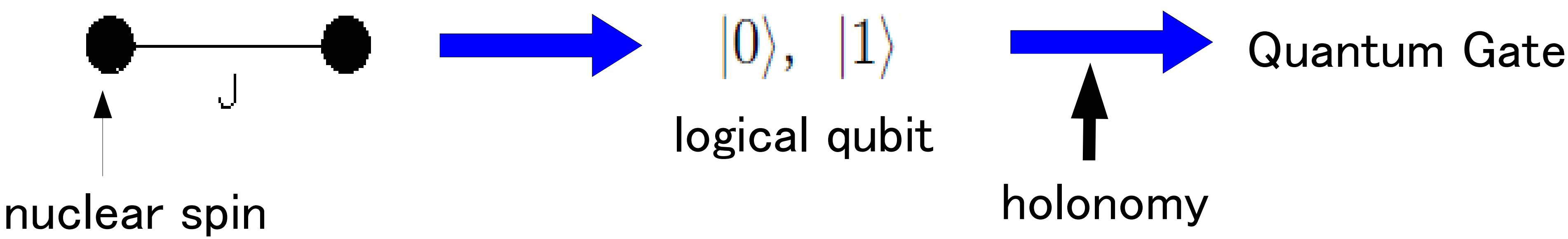
Goal

Implementation of a holonomic quantum computing in a physical system

Holonomic Quantum Computing $\Rightarrow$

- interesting mathematically
- expected to be robust against noise

- With a spin-chain model (proposal with a realistic system)
  - Karimipour and Majd, Phys. Rev. A **72**, 052305 (2005).

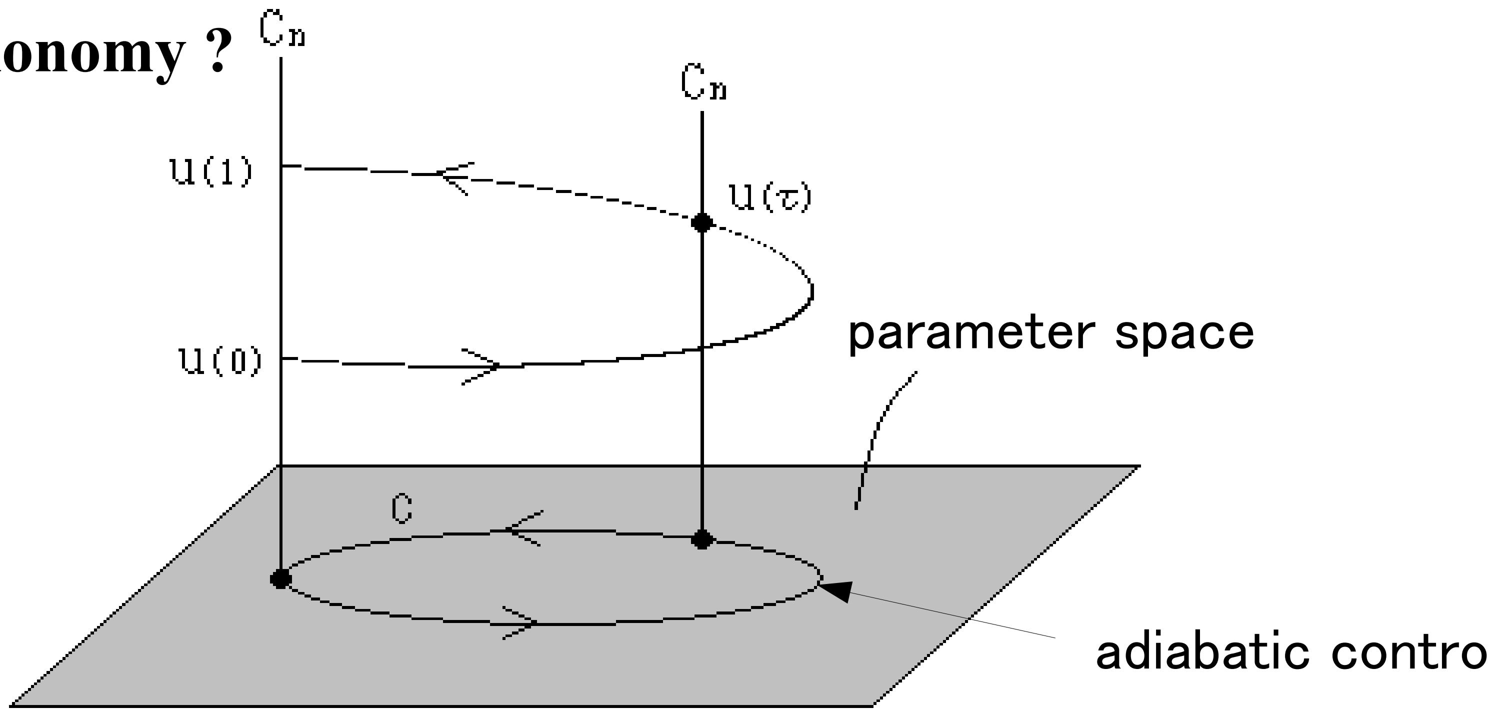


Interaction : Heisenberg-type $\longrightarrow$ Other type of interaction ??

In this work : with Ising-type interaction, Liquid-state NMR in mind !

Holonomic Quantum Computing

- **What is holonomy ?**



$$u(1) = u(0) \cdot \Gamma[C]$$

Γ : Holonomy associated with anti-Hermitian connection

- **What is Holonomic Quantum Computing ?**

⇨ Quantum computing using holonomy

P. Zanardi and M. Rasetti, Phys. Lett. A **264**, 94 (1999).

Isospectral deformation of Hamiltonian

Isospectrally deforming Hamiltonian

Time-dependent Hamiltonian as follows

$$H(t/T) = e^{Xt/T} H_0 e^{-Xt/T} \qquad (0 \leq t \leq T)$$

How to choose X

- It is an anti-Hermitian operator.
- At t = T, the Hamiltonian returns to the initial one.

$$e^X = \mathbb{1}$$

- No undesired transitions into the non-coding space exist (described later).

Holonomy associated with anti-Hermitian connection

adiabatic approximation

holonomy

$$\mathcal{T}\left[\exp(-i\int_0^T H(t/T)\,dt)\right]P_0 \approx e^{-i\int_0^T E_0(t/T)dt}\,\Gamma P_0$$

$$P_0 = \sum_{i=1}^{g} |E_0,\,i\rangle\langle E_0,\,i|$$: projection operator onto the subspace spaned by the ground states

$$\boxed{H(t/T) = e^{Xt/T} H_0 e^{-Xt/T} \qquad \text{Isospectral deformation}}$$

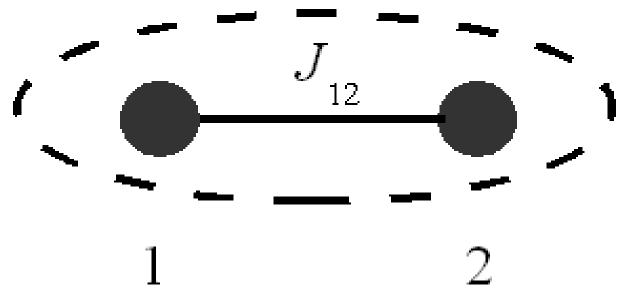
$A_{ij}(\tau) = A_{ij} = \langle E_0,\,i|\,X\,|E_0,\,j\rangle$: (element of) anti-Hermitian connection

$$\Longrightarrow \quad \Gamma = \mathcal{T}\left[\exp(-\int_0^1 A(\tau)d\tau)\right] = e^{-A}$$

The time–evolution operator is determined by A.

Hamiltonian

Hamiltonian of two-spin system with Ising-type interaction



- Hamiltonian

$$H_0 = -\omega\sigma_z \otimes I - \omega I \otimes \sigma_z + J\sigma_z \otimes \sigma_z$$

$$|T_+\rangle = |{+}{+}\rangle \;,\; |T_0\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle + |{-}{+}\rangle) \;,\; |T_-\rangle = |{-}{-}\rangle \;,\; |S_0\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle - |{-}{+}\rangle)$$

$$\begin{aligned} H_0\,|T_+\rangle &= (-2\omega + J)\,|T_+\rangle \\ H_0\,|T_0\rangle &= -J\,|T_0\rangle \\ H_0\,|T_-\rangle &= (2\omega + J)\,|T_-\rangle \\ H_0\,|S_0\rangle &= -J\,|S_0\rangle \end{aligned} \xrightarrow{\omega = J}$$

3-fold degenerate

$|T_+\rangle \;,\; |T_0\rangle \;,\; |S_0\rangle$

Ground states

Result : One-Qubit Quantum Gate

$$X = i\boldsymbol{n}\cdot(\Omega_1\boldsymbol{\sigma}_1 + \Omega_2\boldsymbol{\sigma}_2)$$

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}\otimes I \;,\quad \boldsymbol{\sigma}_2 = I\otimes\boldsymbol{\sigma}$$

$$\mathbf{n} = (n_x\;,\;n_y\;,\;n_z)$$

$$\boldsymbol{\sigma} = (\sigma_x\;,\;\sigma_y\;,\;\sigma_z)\quad,\quad |\mathbf{n}| = 1$$

taking $\Omega_1 = \Omega_2 = \nu$

$$A = \begin{array}{c} \begin{array}{ccc} |T_+\rangle & |T_0\rangle & |S_0\rangle \end{array} \\ \begin{pmatrix} 2\,i\nu\, n_z & \sqrt{2}\,i\,\nu\,(n_x - in_y) & 0 \\ \sqrt{2}\,i\,\nu(n_x + in_y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

There is no transition between

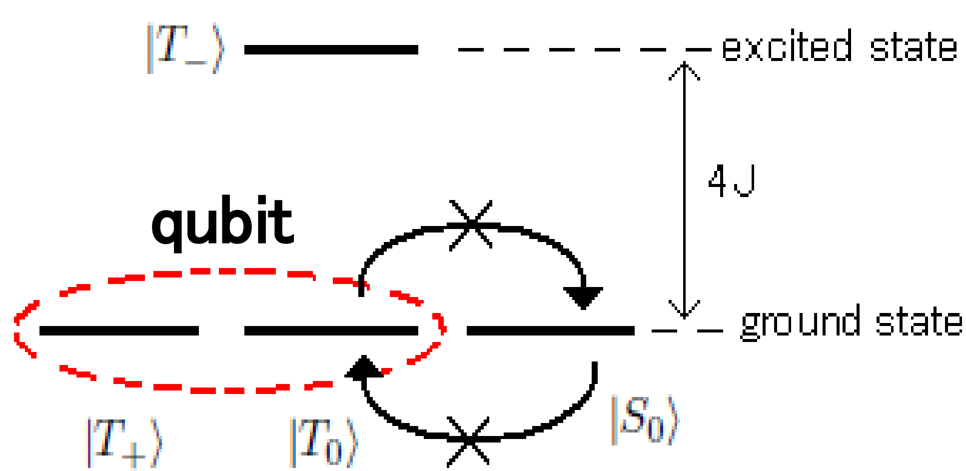
$|T_+\rangle,\ |T_0\rangle$ and $|S_0\rangle$

$$\Downarrow$$

$$|T_+\rangle, |T_0\rangle \longrightarrow |0\rangle, |1\rangle$$

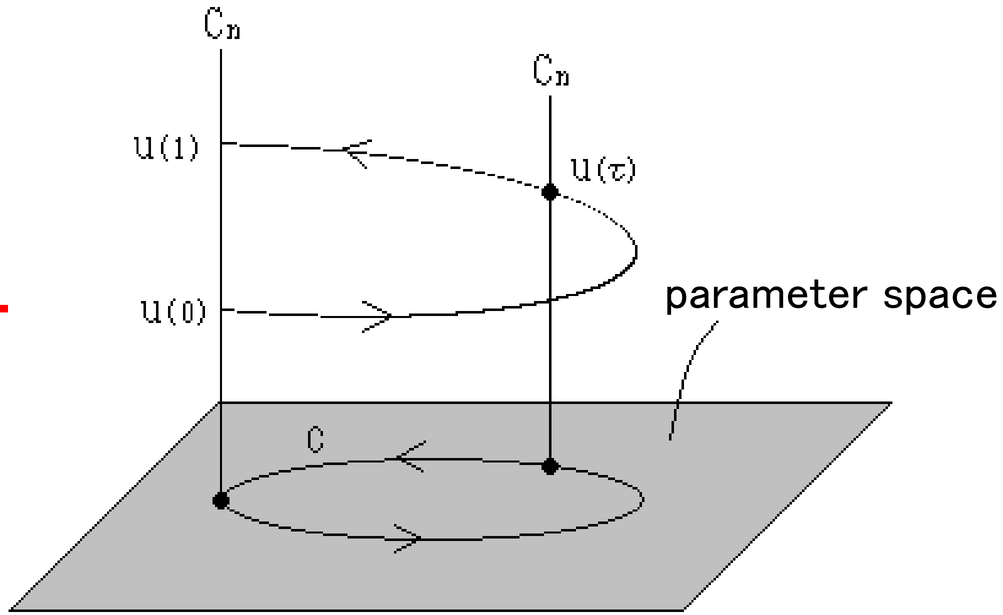
$$A\downarrow_{(T_+,T_0)} = i \begin{array}{c} \begin{array}{cc} |T_+\rangle & |T_0\rangle \end{array} \\ \begin{pmatrix} 2\,\nu\, n_z & \sqrt{2}\,\nu\,(n_x - in_y) \\ \sqrt{2}\,\nu\,(n_x + in_y) & 0 \end{pmatrix} \end{array}$$

$$= \quad i[\nu\, n_z(I + \sigma_z) + \sqrt{2}\,\nu(n_x\,\sigma_x + n_y\,\sigma_y)]$$



taking $\nu = \kappa \quad \pi \quad (\kappa \in \mathbb{N})$

closed loop in the parameter space



$$\Gamma = e^{-i\kappa\pi n_z} e^{-i\kappa\pi\sqrt{2-n_z^2}\,\boldsymbol{m}\cdot\boldsymbol{\sigma}}$$

$$= e^{-i\pi n_z}\begin{pmatrix} \cos\theta - i m_z \sin\theta & -(i m_x + m_y)\sin\theta \\ (-i m_x + m_y)\sin\theta & \cos\theta - i m_z \sin\theta \end{pmatrix}$$

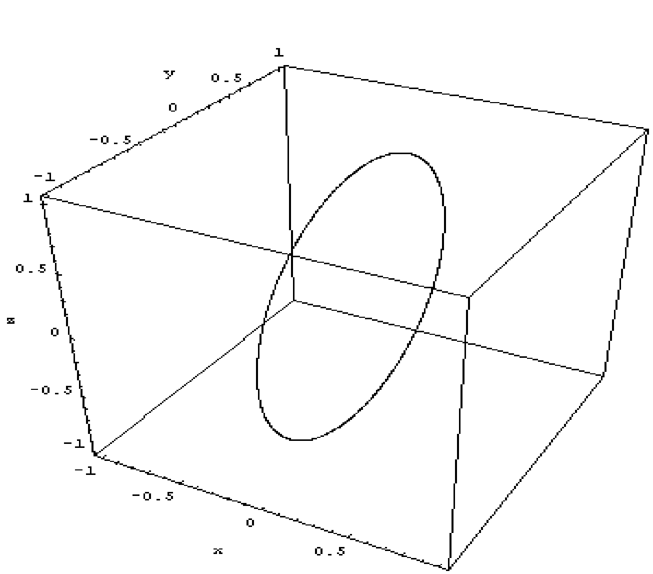
$$\boxed{\theta = \kappa\pi\sqrt{2-n_z^2} \quad , \quad \boldsymbol{m} = \left(\frac{\sqrt{2}n_x}{\sqrt{2-n_z^2}} , \frac{\sqrt{2}n_y}{\sqrt{2-n_z^2}} , \frac{n_z}{\sqrt{2-n_z^2}}\right) \quad , \quad |\boldsymbol{m}| = 1}$$

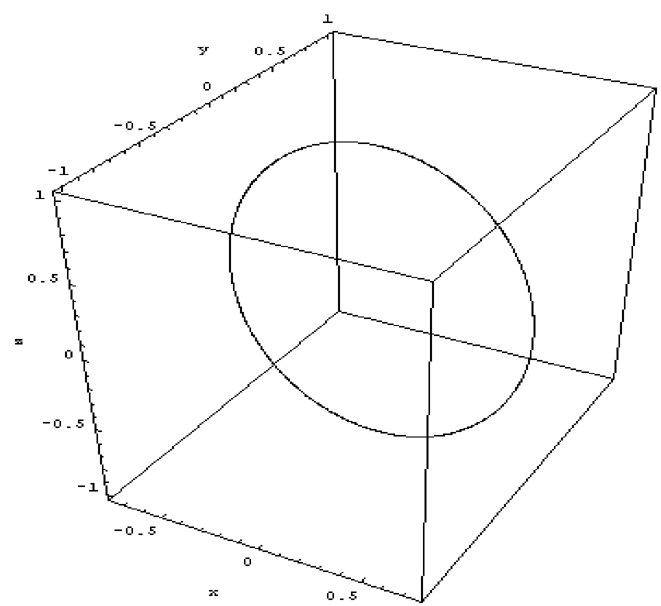
Construction of Quantum Gates

Example

taking $\quad \boldsymbol{n}_1 = (1\,,0\,,0) \quad , \quad \boldsymbol{n}_2 = (0\,,1\,,0) \quad , \quad \kappa = 1 \quad ,$

$$\Gamma_1 = \begin{pmatrix} \cos(\sqrt{2}\pi) & -i\sin(\sqrt{2}\pi) \\ -i\sin(\sqrt{2}\pi) & \cos(\sqrt{2}\pi) \end{pmatrix} , \quad \Gamma_2 = \begin{pmatrix} \cos(\sqrt{2}\pi) & -\sin(\sqrt{2}\pi) \\ \sin(\sqrt{2}\pi) & \cos(\sqrt{2}\pi) \end{pmatrix} , \quad [\Gamma_1\,,\,\Gamma_2] \neq 0$$





parameter space

Hadamard gate

taking

$$\boldsymbol{m} = \frac{1}{\sqrt{2}}(1,\,0,\,1)\ , \quad \frac{\theta_H}{\pi} = \underline{\frac{2}{\sqrt{3}}}\kappa \quad , \ \sigma_H = \frac{1}{\sqrt{2}}(\sigma_{Lx} + \sigma_{Lz})$$

irrational

Here, the Hadamard gate could be constructed if the following condition would be satisfied,

$$\frac{\theta_H}{\pi} = \underline{\frac{1}{2}} + 2n\pi$$

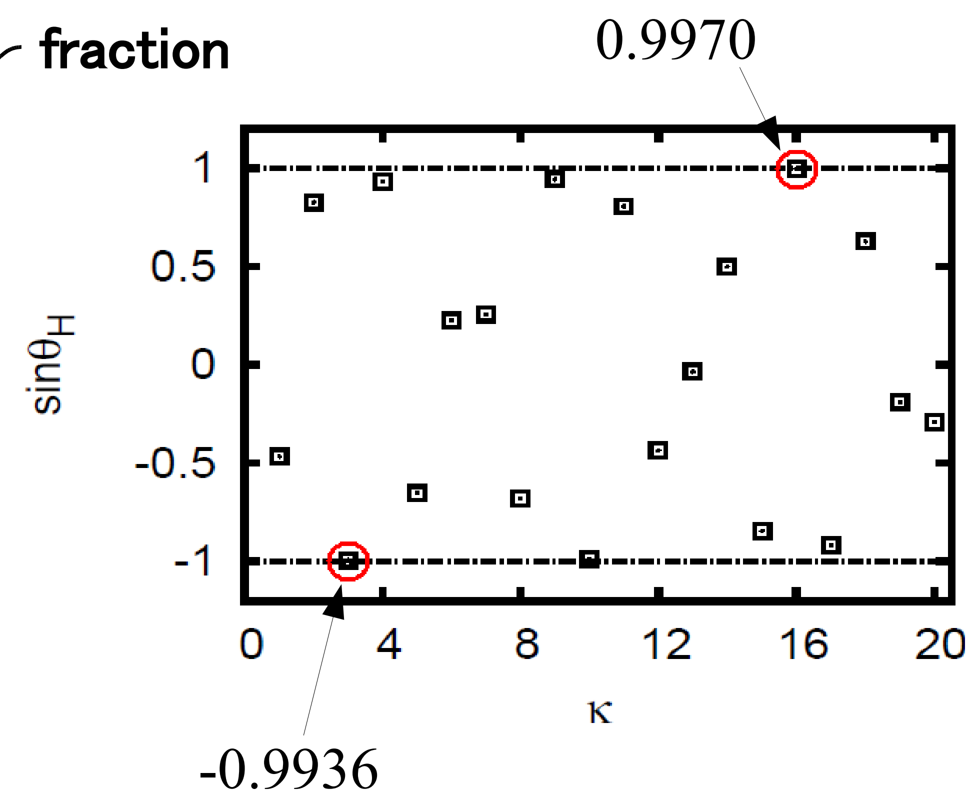
fraction

κ cannot fullfill this condition.

But

We can find κ which satisfies

$$|(\theta - \theta_\kappa) \bmod 2\pi| < \epsilon\,.$$



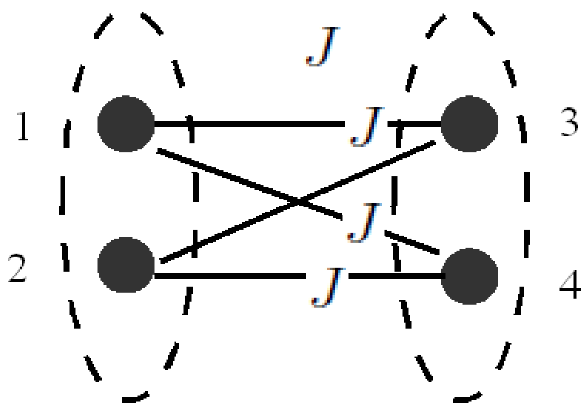
Result 2 : Controlled Gate

$$H_{2D} = H^{(12)} \otimes \mathbb{1}^{(34)} + \mathbb{1}^{(12)} \otimes H^{(34)}$$

$$\boxed{\begin{aligned} H^{(12)} &= -J_{12}\,\sigma_{1z} - J_{12}\,\sigma_{2z} + J_{12}\,\sigma_{1z}\sigma_{2z} \\ H^{(34)} &= -J_{34}\,\sigma_{3z} - J_{34}\,\sigma_{4z} + J_{34}\,\sigma_{3z}\sigma_{4z} \end{aligned}}$$

$$X = X^{(12)} + X^{(34)} + X^{(12)-(34)}$$

$$\boxed{\begin{aligned} X^{(12)} &= i\boldsymbol{n}_{(12)} \cdot (\Omega_{(12)}\boldsymbol{\sigma}_1 + \Omega_{(12)}\boldsymbol{\sigma}_2) \\ X^{(34)} &= i\boldsymbol{n}_{(34)} \cdot (\Omega_{(34)}\boldsymbol{\sigma}_3 + \Omega_{(34)}\boldsymbol{\sigma}_4) \\ X^{(12)-(34)} &= iJ(\sigma_{1z}\sigma_{3z} + \sigma_{1z}\sigma_{4z} + \sigma_{2z}\sigma_{3z} + \sigma_{2z}\sigma_{4z}) \end{aligned}}$$



The Hamiltonian should return to the initial one at the end.

$$\boxed{\begin{gathered} \Omega_{(34)} = \kappa_1\pi \quad (\kappa_1 \in \mathbb{N}) \ , \quad \Omega_{(12)} = \kappa'\pi \quad (\kappa' \in \mathbb{N}) \\ \sqrt{\Omega_{(34)}^2 + 8J^2} = \kappa_2\pi \quad (\kappa_2 \in \mathbb{N}) \end{gathered}}$$

$$\kappa_2 > \kappa_1 \ , \ 3\kappa_1 > \kappa_2$$

$$\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \ , \ 2J = \frac{\pi}{\sqrt{2}}\sqrt{\kappa_2^2 - \kappa_1^2}$$

The obtained gate is

$$\Gamma^{(2)}(J, \kappa') = (-1)^{\kappa'}\,\Gamma^L(J, \kappa')\,\Gamma^C(J), \quad \text{where}$$

$$\Gamma^L(J, \kappa') = e^{-i(\kappa'\pi + J)\sigma_{Lz}} \otimes e^{-i\nu\boldsymbol{k}\cdot\boldsymbol{\sigma}_L} \ , \ \Gamma^C(J) = \underline{|0\rangle_L\langle 0| \otimes I_L + |1\rangle_L\langle 1| \otimes e^{i2J\sigma_{Lz}}} \ .$$

Summary

Two-spin system
(Ising-type interaction)

Construct the logical qubit

Analytical calculation of the holonomy

- Two one-qubit gates that are non-commutable with each other
- Controlled-Z-rotation gate
- Approximate Hadamard gate

are be constructed.