Robust Quantum Gates

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近畿大学大学院 総合理工学研究科 中原研究室 D1

- 近畿大学 理工学部

  卒業論文「ホロノミック量子計算の提案：ダイマー鎖モデル」

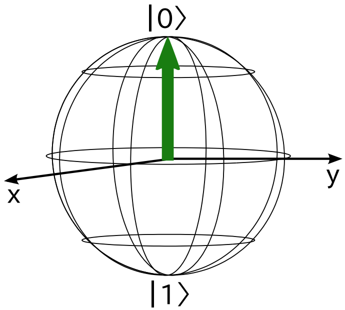
- 大阪大学大学院 理学研究科 博士前期過程

  修士論文「フォノン散乱を考慮したゼーベック係数の第一原理計算手法の開発」

Introduction

Classical Computer

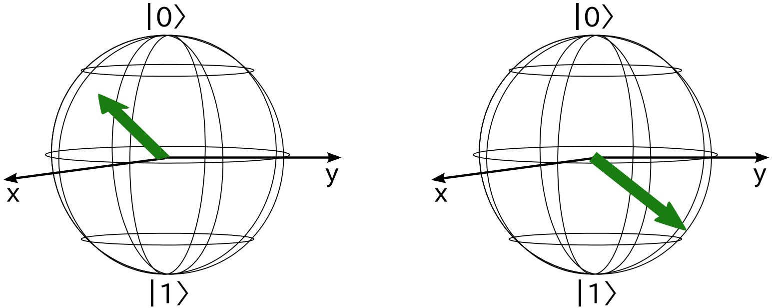
state: $0,\ 1$ (discrete)



Introduction

Quantum Computer

state: $|0\rangle\,,\ |1\rangle\,,\ \alpha\,|0\rangle + \beta\,|1\rangle$ (continuous)

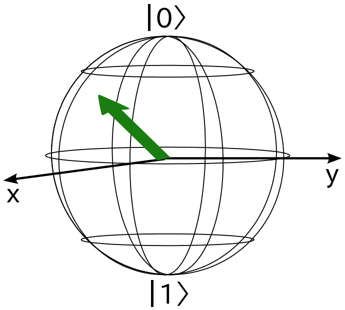


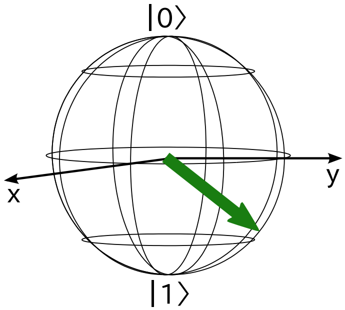
Introduction

Quantum Computer

state: $|0\rangle\,,\ |1\rangle\,,\ \alpha\,|0\rangle + \beta\,|1\rangle$ (continuous)

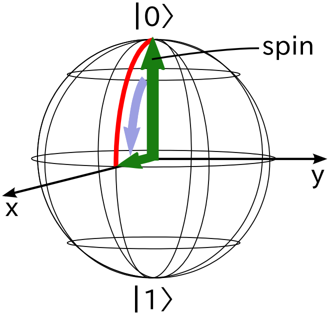
susceptible to error

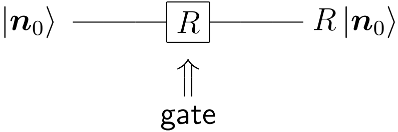




Quantum gate

simple one qubit gate





$$|\boldsymbol{n}_0\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1\ , \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

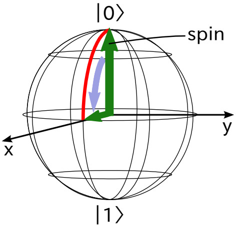
Quantum gate

$$R(\boldsymbol{m}, \theta) = \exp\left(-i\theta \frac{\boldsymbol{m} \cdot \boldsymbol{\sigma}}{2}\right)$$

θ: control field strength $\times$ time

$\boldsymbol{m}$: unit vector $(\boldsymbol{m} \in \mathbb{R}^3)$, $\boldsymbol{\sigma} = (\sigma_x,\ \sigma_y,\ \sigma_z)$

e.g.



$\theta = \pi/2\,,\ \boldsymbol{m} = (0,\ 1,\ 0)$

rotate $\pi/2$ around y axis

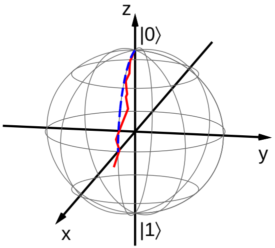
$$R\,|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

Error?

unwanted inputs

- unwanted random inputs

  $\Longrightarrow$ noise



- unwanted systematic inputs

  $\Longrightarrow$ error

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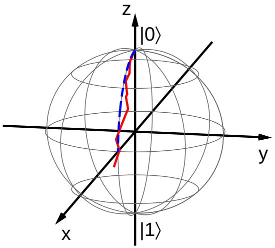

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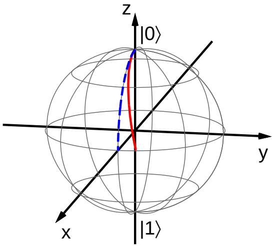

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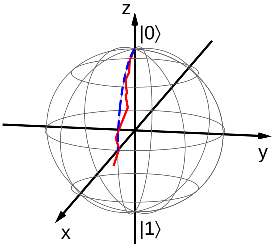
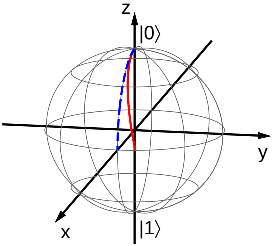


- unwanted systematic inputs

  $\Longrightarrow$ <u>error</u>



in NMR...

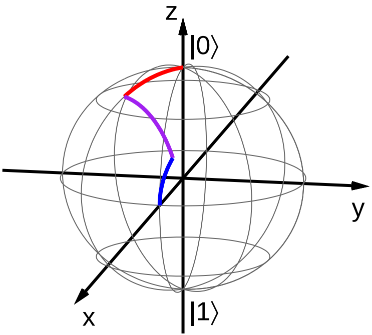
composite pulse

$$|\boldsymbol{n}_0\rangle \; — \boxed{R_1} — \boxed{R_2} — \boxed{R_3} — \cdots — \boxed{R_N} — \quad R_N \cdots R_2 R_1 |\boldsymbol{n}_0\rangle$$

simple pulse

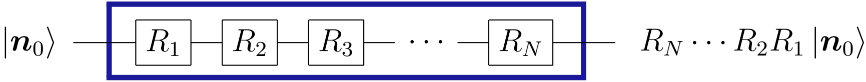
$$|\boldsymbol{n}_0\rangle \; — \boxed{R} — \quad R\,|\boldsymbol{n}_0\rangle$$

$$\boxed{\mathcal{T} \prod_{j=1}^{N} R_j = R}$$

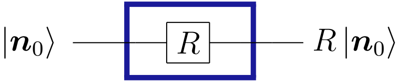


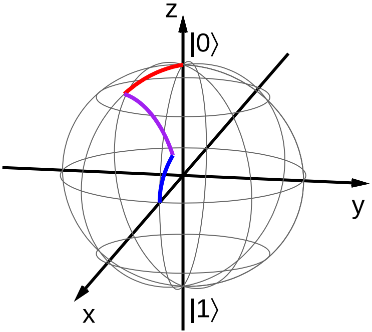
in NMR...

composite pulse



simple pulse





$$\mathcal{T}\prod_{j=1}^{N} R_j = R$$

Framework

How have composite quantum gates been designed up to now?

- from experience
- calculation by using quaternion

  H. K. Cummins, *et. al*, *Phys. Rev.* A **67**, 042308 (2003),
  W. G. Alway, J. A. Jones, *J. Magn. Reson.* 189 (2007) 114-120

  – Not easy to understand physical meaning
  – Complex calculation

$\Downarrow$

- New framework
  + Clear physical meaning
  + Simple calculation

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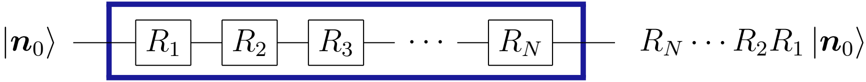
Time-evolution operator

composite pulse



$$R = \exp\left(-i\theta_N \frac{\boldsymbol{m}_N \cdot \boldsymbol{\sigma}}{2}\right) \exp\left(-i\theta_{N-1} \frac{\boldsymbol{m}_{N-1} \cdot \boldsymbol{\sigma}}{2}\right) \cdots$$

$$\Downarrow$$

$$U_\lambda(1,0) := \mathcal{T} \exp\left(-i \int_0^1 dt H(\lambda(t))\right)$$

Time-evolution operator

$$U_\lambda(1,0) \in SU(n)$$

$$U_\lambda(1,0) := \mathcal{T}\exp\left(-i\int_0^1 dt H(\lambda(t))\right)$$

Hamiltonian $H(\lambda(t)) := \lambda_\mu(t)\tau_\mu$

in case of $n=2$

$$H(\lambda(t)) = \lambda_\mu(t)\frac{\sigma_\mu}{2}\,, \quad (\sigma_\mu : \text{ Pauli matrices})$$

control parameter $\lambda(t) = (\lambda_1(t), \ldots, \lambda_{n^2-1}(t))$

n^2-1 dimension orthogonal basis $\tau_\mu \quad (\mu = 1, \ldots, n^2-1)$

Time-evolution operator

$$\boxed{U_\lambda(1,0) \in SU(n)}$$

$$U_\lambda(1,0) := \mathcal{T} \exp\left(-i \int_0^1 dt H(\lambda(t))\right)$$

e.g.

$$n = 2\ , \quad \lambda = (\theta,\ 0,\ 0)$$

$$U_\lambda(1,0) = \exp\left(-i\theta\,\frac{\sigma_x}{2}\right) = R(\boldsymbol{x}, \theta)$$

Hamiltonian $H(\lambda(t)) := \lambda_\mu(t)\tau_\mu$

Definition of “robust quantum gate”

time-evolution operator with error

$$U_{\lambda+\delta\lambda}(1,0) = U_\lambda(1,0)\Big(1 + \underline{\mathcal{O}(|\delta\lambda|)}\Big)$$

if the first order term of error vanishes

$\Longrightarrow$ robust against error

$$U_{\lambda+\delta\lambda}(1,0) = U_\lambda(1,0)\left(1 + \underline{\mathcal{O}(|\delta\lambda|^2)}\right)$$

$\delta\lambda(t)$: error ($|\delta\lambda(t)| \ll |\lambda(t)|$)

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$\delta\lambda(t)$: error ($|\delta\lambda(t)| \ll |\lambda(t)|$)

time-evolution with error

time-evolution operator (with error)

$$U_{\lambda+\delta\lambda}(1,0) = U_\lambda(1,0) - \underline{iU_\lambda(1,0)\int_0^1 dt\, H_I(\delta\lambda(t))} + \mathcal{O}(|\delta\lambda|^2)$$

$H_I(\delta\lambda(t))$: error term of Hamiltonian at interaction picture

robustness condition

$$\int_0^1 dt\, H_I(\delta\lambda(t)) = 0$$

time-evolution with error

time-evolution operator (with error)

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$H_I(\delta\lambda(t))$: error term of Hamiltonian at interaction picture

robustness condition

$$\int_0^1 dt\, H_I(\delta\lambda(t)) = 0$$

classification of errors

systematic error: $\delta\lambda_\mu(t) = F_\mu(\lambda(t)) = f_\mu + f_{\mu\nu}\lambda_\nu(t) + \dots$

$\Downarrow$

robustness condition

$$\int_0^1 dt\, H_I(\delta\lambda(t)) = 0$$

$$f_\mu \int_0^1 dt \tilde{\tau}_\mu(t) \; + \; f_{\mu\nu} \int_0^1 dt \tilde{\tau}_\mu(t)\lambda_\nu(t) \; + \; \dots = 0$$

$$H_I(\delta\lambda(t)) = \delta\lambda_\mu(t)\tilde{\tau}_\mu(t)$$

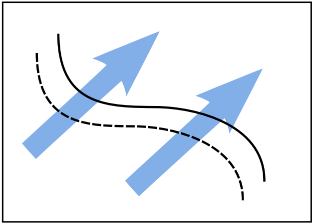
$$\tilde{\tau}_\mu(t) = U_\lambda(t)^\dagger \tau_\mu U_\lambda(t) \; , \qquad H(\delta\lambda(t)) = \delta\lambda_\mu \tau_\mu$$

$$f_\mu \int_0^1 dt\, \tilde{\tau}_\mu(t) = 0$$

$$f_{\mu\nu} \int_0^1 dt\, \tilde{\tau}_\mu(t)\lambda_\nu(t) = 0$$

expectation value: $\langle\varphi|\, \tilde{\tau}_\mu(t)\, |\varphi\rangle = \langle\varphi(t)|\, \tau_\mu\, |\varphi(t)\rangle = \varphi_\mu(t)$

$$\int_0^1 dt\, \boldsymbol{\varphi}(t) = 0$$



$\mathbb{R}^{n^2-1}$: control parameter $\lambda(t)$

$\boldsymbol{\varphi}(t)$: generalized Bloch vector

$$f_\mu \int_0^1 dt\, \tilde{\tau}_\mu(t) = 0$$

$$\boxed{f_{\mu\nu} \int_0^1 dt\, \tilde{\tau}_\mu(t) \lambda_\nu(t) = 0}$$

$$f_{\mu\nu} = \underline{\frac{1}{N}\delta_{\mu\nu} f_{\rho\rho}} + \underline{\frac{1}{2}(f_{\mu\nu} - f_{\nu\mu})} + \underline{\left[\frac{1}{2}(f_{\mu\nu} + f_{\nu\mu}) - \frac{1}{N}\delta_{\mu\nu} f_{\rho\rho}\right]}$$

$$f_\mu \int_0^1 dt\, \tilde{\tau}_\mu(t) = 0$$

$$f_{\mu\nu} \int_0^1 dt\, \tilde{\tau}_\mu(t)\lambda_\nu(t) = 0$$

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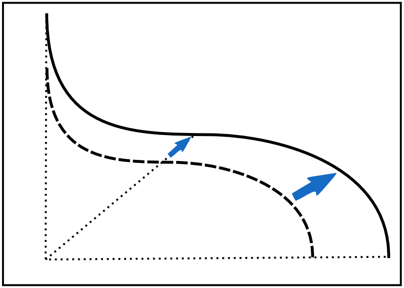
$$\int_0^1 dt\, \tilde{\tau}_\mu \lambda_\mu = 0$$

$\Longrightarrow$ error on norm $|\lambda(t)|$



$\mathbb{R}^{n^2-1}$: control parameter $\lambda(t)$ space

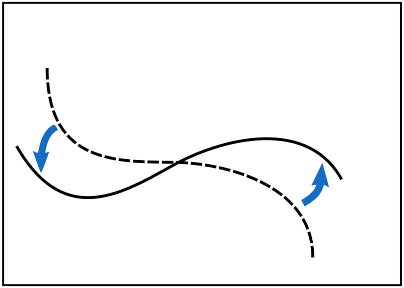
$$f_\mu \int_0^1 dt\, \tilde{\tau}_\mu(t) = 0$$

$$f_{\mu\nu} \int_0^1 dt\, \tilde{\tau}_\mu(t)\lambda_\nu(t) = 0$$

$$f_{\mu\nu} = \frac{1}{N}\delta_{\mu\nu} f_{\rho\rho} + \frac{1}{2}(f_{\mu\nu} - f_{\nu\mu}) + \left[\frac{1}{2}(f_{\mu\nu} + f_{\nu\mu}) - \frac{1}{N}\delta_{\mu\nu} f_{\rho\rho}\right]$$

$$\int_0^1 dt\, (\tilde{\tau}_\mu\lambda_\nu - \tilde{\tau}_\nu\lambda_\mu) = 0$$

$\Longrightarrow$ rotation of $\lambda(t)$



$\mathbb{R}^{n^2-1}$: control parameter $\lambda(t)$ space

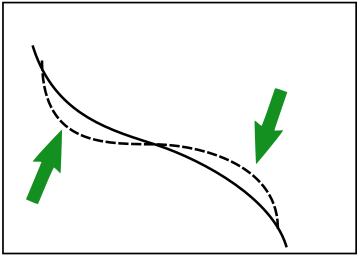
$$f_\mu \int_0^1 dt\, \tilde{\tau}_\mu(t) = 0$$

$$f_{\mu\nu} \int_0^1 dt\, \tilde{\tau}_\mu(t)\lambda_\nu(t) = 0$$

$$f_{\mu\nu} = \frac{1}{N}\delta_{\mu\nu} f_{\rho\rho} + \frac{1}{2}(f_{\mu\nu} - f_{\nu\mu}) + \left[\frac{1}{2}(f_{\mu\nu} + f_{\nu\mu}) - \frac{1}{N}\delta_{\mu\nu} f_{\rho\rho}\right]$$

$$\int_0^1 dt \left[\frac{\tilde{\tau}_\mu\lambda_\nu + \tilde{\tau}_\nu\lambda_\mu}{2} - \frac{\delta_{\mu\nu}\tilde{\tau}_\rho\lambda_\rho}{N}\right] = 0$$

$\Longrightarrow$ torsion of $\lambda(t)$



$\mathbb{R}^{n^2-1}$: control parameter $\lambda(t)$ space

$$f_\mu \int_0^1 dt\, \tilde{\tau}_\mu(t) = 0$$

$$\int_0^1 dt\, \boldsymbol{\varphi}(t) = 0 \quad \Longrightarrow$$ constant error

$$f_{\mu\nu} \int_0^1 dt\, \tilde{\tau}_\mu(t)\lambda_\nu(t) = 0$$

$$\int_0^1 dt\, \tilde{\tau}_\mu \lambda_\mu = 0 \quad \Longrightarrow$$ norm of $\lambda(t)$

$$\int_0^1 dt\, (\tilde{\tau}_\mu \lambda_\nu - \tilde{\tau}_\nu \lambda_\mu) = 0 \quad \Longrightarrow$$ rotation of $\lambda(t)$

$$\int_0^1 dt\, \left[\frac{1}{2}(\tilde{\tau}_\mu \lambda_\nu + \tilde{\tau}_\nu \lambda_\mu) - \frac{1}{N}\delta_{\mu\nu}\tilde{\tau}_\rho \lambda_\rho\right] = 0 \quad \Longrightarrow$$ torsion of $\lambda(t)$

$$f_{\mu} \int_0^1 dt\, \tilde{\tau}_{\mu}(t) = 0$$

$$\int_0^1 dt\, \varphi(t) = 0 \implies \text{constant error}$$

$$f_{\mu\nu} \int_0^1 dt\, \tilde{\tau}_{\mu}(t)\lambda_{\nu}(t) = 0$$

$$\int_0^1 dt\, \tilde{\tau}_{\mu}\lambda_{\mu} = 0 \implies \text{norm of } \lambda(t)$$

$$\int_0^1 dt\, (\tilde{\tau}_{\mu}\lambda_{\nu} - \tilde{\tau}_{\nu}\lambda_{\mu}) = 0 \implies \text{rotation of } \lambda(t)$$

$$\int_0^1 dt\, \left[\frac{1}{2}(\tilde{\tau}_{\mu}\lambda_{\nu} + \tilde{\tau}_{\nu}\lambda_{\mu}) - \frac{1}{N}\delta_{\mu\nu}\tilde{\tau}_{\rho}\lambda_{\rho}\right] = 0 \implies \text{torsion of } \lambda(t)$$

example

Discretization

discretization

$$U_{\lambda+\delta\lambda}(1,0) = U_{\lambda^N+\delta\lambda^N}(1,t_{N-1})\dots U_{\lambda^1+\delta\lambda^1}(t_1,0)$$

$$U_{\lambda^j}(t_j,t_{j-1}) = R(\boldsymbol{m}_j,\theta_j) = \exp\left(-i\frac{\theta_j}{2}\boldsymbol{m}_j\cdot\boldsymbol{\sigma}\right)$$

pulse strength error

$$U_{\lambda+\delta\lambda}(t_j,t_{j-1}) = \exp\left(-i\theta_j(1+\varepsilon)\frac{\boldsymbol{m}_j\cdot\boldsymbol{\sigma}}{2}\right)$$

error in θ (rotating angle)

Robustness condition

robustness condition

$$\int_0^1 dt\, H_I(\lambda(t)) = 0 \implies \int_0^1 dt\, U_\lambda(t_j, 0)^\dagger H(\lambda(t)) U_\lambda(t_j, 0) = 0$$

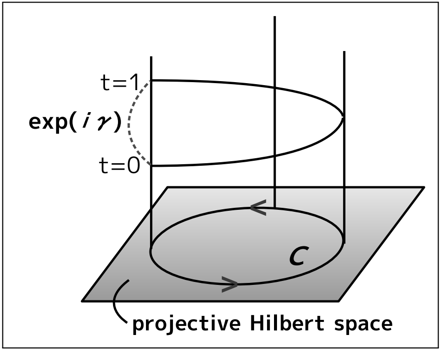
robustness condition for error in θ

$$\sum_{j=1}^{N} U_{\lambda^N} \dots U_{\lambda^{j+1}} (H_j T_j) U_{\lambda^j} \dots U_{\lambda^1} = 0$$

$$H_j = \frac{\theta_j}{2} \boldsymbol{m}_j \cdot \boldsymbol{\sigma} \frac{1}{T_j} \;, \qquad (T_j = t_j - t_{j-1})$$

Phases

Aharonov-Anandan phase



$$|\boldsymbol{n}(1)\rangle = \underline{\exp(i\gamma)}\,|\boldsymbol{n}(0)\rangle$$

$$\gamma = \gamma_\mathrm{d} + \gamma_\mathrm{g}$$

dynamic phase: $\gamma_\mathrm{d} = -\int_0^T \langle \boldsymbol{n}(t)|\, H\, |\boldsymbol{n}(t)\rangle\, dt$

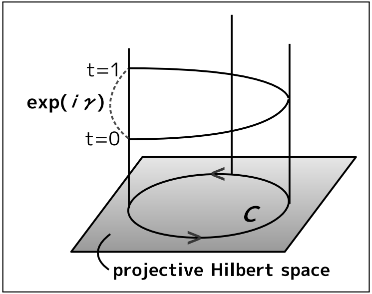
geometric phase: $\gamma_\mathrm{g} = \gamma - \gamma_\mathrm{d}$ $\quad$ (H : Hamiltonian)

Cyclic States

$$|\boldsymbol{n}_+(0)\rangle\,,\quad |\boldsymbol{n}_-(0)\rangle$$

$$\langle \boldsymbol{n}_+(0)|\,\boldsymbol{n}_-(0)\rangle = 0$$

$$|\boldsymbol{n}_\pm(1)\rangle = \exp\left(i\gamma_\pm\right)|\boldsymbol{n}_\pm(0)\rangle$$



$$|\boldsymbol{n}(0)\rangle = a_+\,|\boldsymbol{n}_+(0)\rangle + a_-\,|\boldsymbol{n}_-(0)\rangle$$

$$|\boldsymbol{n}(1)\rangle = a_+e^{i\gamma_+}\,|\boldsymbol{n}_+(0)\rangle + a_-e^{i\gamma_-}\,|\boldsymbol{n}_-(0)\rangle$$

quantum gate(time-evolution operator)

$$U = a_+e^{i\gamma_+}\,|\boldsymbol{n}_+(0)\rangle\,\langle\boldsymbol{n}_+(0)| + a_-e^{i\gamma_-}\,|\boldsymbol{n}_-(0)\rangle\,\langle\boldsymbol{n}_-(0)|$$

phase and robustness condition

expectation value for cyclic states...

$$\langle \boldsymbol{n}(0)| \sum_{j=1}^{N} U_{\lambda^N} \ldots U_{\lambda^{j+1}} (H_j T_j) U_{\lambda^j} \ldots U_{\lambda^1} |\boldsymbol{n}(0)\rangle$$

$$= e^{-i\theta/2} \sum_{j=1}^{N} \langle \boldsymbol{n}(t_j)| H_j T_j |\boldsymbol{n}(t_j)\rangle$$

$$= e^{-i\theta/2} \sum_{j=1}^{N} \gamma_{\mathrm{d},j}$$

$$\sum_{j=1}^{N} \gamma_{\mathrm{d},j} = 0 \quad \Longrightarrow \quad U_{\lambda+\delta\lambda} = \underline{U_\lambda\left(1 + \mathcal{O}(\varepsilon^2)\right)}$$

dynamic phase is 0 $\Longleftrightarrow$ robust against "systematic control field strength error"

phase and robustness condition

expectation value for cyclic states...

$$\langle \boldsymbol{n}(0)| \sum_{j=1}^{N} U_{\lambda^N} \dots U_{\lambda^{j+1}} (H_j T_j) U_{\lambda^j} \dots U_{\lambda^1} |\boldsymbol{n}(0)\rangle$$

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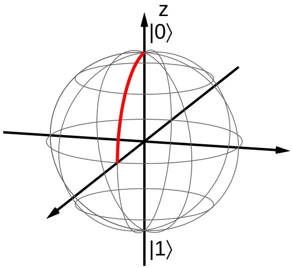
dynamic phase is 0	$\Longleftrightarrow$	robust against "systematic control field strength error"

W1 sequence

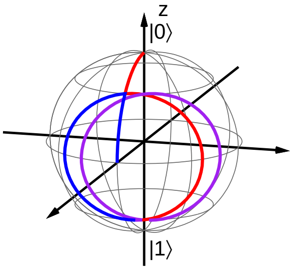
$$U_{\mathrm{W1}} = R(\boldsymbol{m}_1, \pi) R(\boldsymbol{m}_2, 2\pi) R(\boldsymbol{m}_1, \pi) = I$$

Single



$R(\boldsymbol{x}, \pi/2)$

Composite



$R(\boldsymbol{x}, \pi/4)\, U_{\mathrm{W1}}\, R(\boldsymbol{x}, \pi/4)$

$$\phi_1 = \pm \arccos\left(-\theta/(4\pi)\right), \qquad \phi_2 = 3\phi_1$$

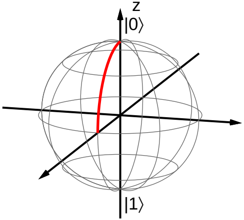
$$\boldsymbol{m}_i = (\cos\phi_i,\ \sin\phi_i,\ 0)$$

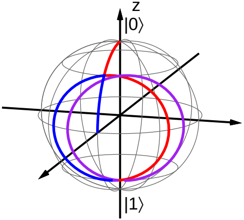
W1 sequence

$$U_{\mathrm{W1}} = e^{i\gamma_{\mathrm{W1}}} |\boldsymbol{x}\rangle\langle\boldsymbol{x}| + e^{-i\gamma_{\mathrm{W1}}} |-\boldsymbol{x}\rangle\langle-\boldsymbol{x}|$$

$$\gamma_{\mathrm{W1}} = \gamma_{\mathrm{g,W1}} + \gamma_{\mathrm{d,W1}} = 0 \ , \quad \underline{\gamma_{\mathrm{d,W1}}} = -\gamma_{\mathrm{g,W1}} = \underline{\theta/2}$$

Dynamic Phase Gate		Geometric Phase Gate
$R(\boldsymbol{x}, \theta)$	$\Longrightarrow$	$R(\boldsymbol{x}, \theta/2) U_{\mathrm{W1}} R(\boldsymbol{x}, \theta/2)$



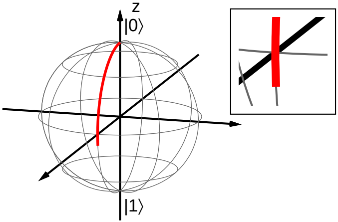


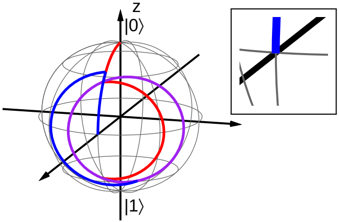
$$\gamma_{\mathrm{d}} = -\theta/2$$

$$\boxed{\gamma_{\mathrm{d}} = -\theta/2 + \underline{\theta/2} = 0}$$

W1 sequence

simple 90x gate | composite 90x gate





simple 90x gate and composite 90x gate with 10% error in θ

magnitude of displacement

$$1 - \frac{1}{2} \sum_{j=0,1} \boldsymbol{n}^j_{\lambda+\delta\lambda} \cdot \boldsymbol{n}^j_{\lambda} = \begin{cases} \sim 10^{-2} & \text{(simple)} \\ \sim 10^{-6} & \text{(composite)} \end{cases}$$

$$\boldsymbol{n}^j_{\lambda} = \langle j | \, U^\dagger_\lambda \boldsymbol{\sigma} U_\lambda \, | j \rangle \ , \quad \boldsymbol{n}^j_{\lambda+\delta\lambda} = \langle j | \, U^\dagger_{\lambda+\delta\lambda} \boldsymbol{\sigma} U_{\lambda+\delta\lambda} \, | j \rangle$$

Summary

- conditions for robust quantum gate
- physical meaning
- phases and robustness

References

- Y. Kondo and M. Bando, accepted for publication in J. Phys. Soc. Jpn., arXiv:1005.3917.